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ON W_3 -MORPHISMS AND THE GEOMETRY OF PLANE CURVES

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ABSTRACT

We provide a description of W_3 transformations in terms of deformations of convex curves in two dimensional Euclidean space. This geometrical interpretation sheds some light on the nature of finite W_3 -morphisms. We also comment on how this construction can be extended to the case of W_n and “nicely curved” curves in \mathbf{R}^{n-1} .

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W -algebras were first introduced by Zamolodchikov in the framework of two dimensional conformal field theory. It was shown in [1] by using the bootstrap method, that the extension of the Virasoro algebra by a single field of spin 3 (W) yielded a non-linear associative algebra, denoted since then by W_3 . The geometrical significance of the Virasoro generator T is well understood:

$$Q_\epsilon = \oint dz \epsilon(z) T(z)$$

is the generator of conformal transformations (or, equivalently, of diffeomorphisms of the circle). It is therefore natural to ask what is the geometrical significance, if any, of the transformation generated by

$$Q_\eta = \oint dz \eta(z) W(z) ?$$

At present, there are several different ways of providing a geometrical interpretation for these W_3 transformations (W_3 -morphisms) in their classical limit, *i.e.* operator product expansions, or equivalently commutators, are substituted by Poisson brackets [2]. Most of them rely on the extrinsic geometry of curves and surfaces [3], although different approaches based on intrinsic geometry have also proved fruitful [4].

The purpose of this note is to give yet another interpretation of W_3 -morphisms in terms of the extrinsic geometry of curves in a particularly simple and, we believe, enlightening setting.

Our main result can be summarized as follows: *A W_3 -morphism can be defined as a map between strictly convex curves in \mathbf{R}^2 .*

The remainder of the paper will be dedicated to justifying the above statement. Before proceeding, however, it will be necessary to introduce some well-known results on the geometry of plane curves.

The geometry of closed plane curves

Our basic object of study will be the geometry of parametrized closed curves

$$\begin{aligned} \gamma : [a, b] &\rightarrow \mathbf{R}^2 \\ t &\mapsto \mathbf{x}(t), \end{aligned}$$

which are immersions, *i.e.* which satisfy $\dot{\mathbf{x}}(t) = d\mathbf{x}/dt \neq 0$ for all t . In what follows, it will be useful to consider arc-length parametrized curves. The arc-length function $s : [a, b] \rightarrow \mathbf{R}$ is defined as usual by

$$s(t) = \int_a^t |\dot{\mathbf{x}}(t')| dt'.$$

A tangent vector field to γ for all t is given by its velocity vector. Its modulus e is the one-dimensional induced *einbein*. We can now introduce the normalized tangent vector \mathbf{v}_1 ,

where

$$\mathbf{v}_1 = \frac{d\mathbf{x}}{ds} = \frac{\dot{\mathbf{x}}}{e}.$$

The curvature function can now be defined as the modulus of its derivative,

$$\frac{d\mathbf{v}_1}{ds} = \kappa \mathbf{v}_2 = \frac{\ddot{\mathbf{x}}_\perp}{e^2},$$

where the unit vector \mathbf{v}_2 is by construction orthogonal to \mathbf{v}_1 . We can still go a little further and define the signed curvature $\tilde{\kappa}$ using the natural orientation in \mathbf{R}^2 , *i.e.* $\tilde{\kappa} = \kappa$ if the frame $(\mathbf{v}_1, \mathbf{v}_2)$ is positively oriented, and $\tilde{\kappa} = -\kappa$ otherwise.

The geometrical relevance of the induced metric and the signed curvature function is revealed by the fact that any two parametrized curves in \mathbf{R}^2 with the same values of e and $\tilde{\kappa}$ should be related by an Euclidean motion, *i.e.* translation and rotation.

A curve is called simple if it is one-to-one. Among the simple closed curves we may distinguish the convex ones. They are defined by the property that they are the boundary of a convex set¹. But they are more easily characterized for our purposes by the fact that they are simple closed curves with $\kappa \geq 0$. We will work with “strictly convex” curves for which the curvature function is everywhere strictly greater than zero (these curves are also known as ovals in the mathematical literature).

Infinitesimal deformations of strictly convex curves and the W_3 -algebra

An infinitesimal transformation of a curve γ is completely determined by a vector field with support on γ . The importance of working with strictly convex curves comes from the fact that $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}_\perp$ provide us with an orthogonal frame for the tangent space $T_p \mathbf{R}^2$ for all $p \in \gamma$. Explicitly,

$$\delta_{\alpha, \eta} \mathbf{x} = \alpha(t) \dot{\mathbf{x}} + \eta(t) \ddot{\mathbf{x}}_\perp.$$

In order to study how the above transformations act on γ it is enough to study their action on e and κ . They are given by

$$\begin{aligned} \delta_{\alpha, \eta} e &= \dot{\alpha}e + \alpha\dot{e} - \eta\kappa^2 e^3, \\ \delta_{\alpha, \eta} \kappa &= \alpha\dot{\kappa} + \dot{\eta}\kappa + \dot{\eta} \left(3\frac{\dot{e}\kappa}{e} + 2\dot{\kappa} \right) + \eta \left(\ddot{\kappa} + 3\frac{\dot{e}\dot{\kappa}}{e} + 2\frac{\ddot{e}\kappa}{e} + \kappa^3 e^2 \right), \end{aligned}$$

where we have used the two-dimensional trivial identity

$$\ddot{\mathbf{x}} = \frac{(\ddot{\mathbf{x}} \ddot{\mathbf{x}}_\perp)}{\ddot{\mathbf{x}}_\perp^2} \ddot{\mathbf{x}}_\perp + \frac{(\ddot{\mathbf{x}} \dot{\mathbf{x}})}{\dot{\mathbf{x}}^2} \dot{\mathbf{x}},$$

which can also be written in terms of e and κ as

$$\ddot{\mathbf{x}} = \left(\frac{\dot{\kappa}}{\kappa} + 3\frac{\dot{e}}{e} \right) \ddot{\mathbf{x}}_\perp + \left(\frac{\ddot{e}}{e} - \kappa^2 e^2 \right) \dot{\mathbf{x}}.$$

If we now compute the commutator of two such transformations we find that, in general, they close among themselves with structure constants that are functions of e and κ . It is

¹ Any subset of \mathbf{R}^2 is called convex if the line segment joining any two points in the set belongs to that set.

possible however to choose a parametrization such that the algebra has a particularly simple form. Let us consider

$$\begin{aligned}\delta_\epsilon \mathbf{x} &= \epsilon \dot{\mathbf{x}}, \\ \delta_\rho \mathbf{x} &= \rho \ddot{\mathbf{x}}_\perp - \left(\frac{1}{2} \dot{\rho} + \rho \left(\frac{\dot{e}}{e} + \frac{2 \dot{\kappa}}{3 \kappa} \right) \right) \dot{\mathbf{x}}.\end{aligned}$$

A long but straightforward computation now yields

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}] &= \delta_\epsilon, \quad \text{where } \epsilon = \epsilon_1 \dot{\epsilon}_2 - \dot{\epsilon}_1 \epsilon_2, \\ [\delta_\epsilon, \delta_\rho] &= \delta_{\tilde{\rho}}, \quad \text{where } \tilde{\rho} = \epsilon \dot{\rho} - 2 \dot{\epsilon} \rho, \\ [\delta_{\rho_1}, \delta_{\rho_2}] &= \delta_\epsilon, \quad \text{where } \epsilon = \frac{2}{3} \rho_1 \dot{\rho}_2 T - \frac{1}{4} \dot{\rho}_1 \ddot{\rho}_2 + \frac{1}{6} \rho_1 \ddot{\rho}_2 - (\rho_1 \leftrightarrow \rho_2),\end{aligned}$$

which as we will see is nothing but the algebra of infinitesimal W_3 transformations with “energy-momentum tensor” T given by

$$T = 2 \frac{\ddot{e}}{e} - 3 \frac{\dot{e}^2}{e^2} - \frac{\dot{e} \dot{\kappa}}{e \kappa} + \frac{\ddot{\kappa}}{\kappa} - \frac{4}{3} \frac{\dot{\kappa}^2}{\kappa^2} + e^2 \kappa^2.$$

In order that the infinitesimal transformations close when acting on T we have to introduce its “ W -partner”

$$\begin{aligned}W = & - \frac{1}{6} \frac{\ddot{\kappa}}{\kappa} + \frac{5}{6} \frac{\dot{\kappa} \ddot{\kappa}}{\kappa^2} - \frac{20}{27} \frac{\dot{\kappa}^3}{\kappa^3} - \frac{2}{3} \kappa \dot{\kappa} e^2 \\ & - \frac{5}{6} \frac{\dot{\kappa}^2 \dot{e}}{\kappa^2 e} - \frac{1}{2} \frac{\kappa \dot{e}^2}{\kappa e^2} + \frac{1}{2} \frac{\dot{\kappa} \dot{e}}{\kappa e} + \frac{1}{6} \frac{\dot{\kappa} \ddot{e}}{\kappa e}.\end{aligned}$$

Then the infinitesimal transformations on T and W read

$$\begin{aligned}\delta_\epsilon T &= 2 \ddot{\epsilon} + 2 \dot{\epsilon} T + \epsilon \dot{T}, \\ \delta_\epsilon W &= 3 \dot{\epsilon} W + \epsilon \dot{W}, \\ \delta_\rho T &= 3 \dot{\rho} W + 2 \rho \dot{W}, \\ \delta_\rho W &= - \frac{1}{6} \rho^{(v)} - \frac{5}{6} \ddot{\rho} T - \frac{5}{4} \ddot{\rho} \dot{T} - \dot{\rho} \left(\frac{3}{4} \ddot{T} + \frac{2}{3} T^2 \right) - \rho \left(\frac{1}{6} \ddot{T} + \frac{2}{3} T \dot{T} \right)\end{aligned}$$

which are nothing but the standard classical limit of Zamolodchikov’s W_3 -algebra [2].

We have just shown that infinitesimal W_3 -morphisms can be understood as infinitesimal deformations of strictly convex curves in \mathbf{R}^2 , therefore it follows that a finite W_3 transformation can be given a natural geometrical realization as a map between two strictly convex plane curves.

W_n and nicely curved curves in \mathbf{R}^{n-1}

It should be clear by now how the preceding techniques may be extended to the case of W_n . Any curve γ in \mathbf{R}^{n-1} satisfies the generalized Frenet equations

$$\begin{aligned}\frac{d\mathbf{v}_1}{ds} &= \kappa_1 \mathbf{v}_2, \\ \frac{d\mathbf{v}_j}{ds} &= \kappa_j \mathbf{v}_{j+1} - \kappa_{j-1} \mathbf{v}_{j-1}, \quad 1 < j < n-1, \\ \frac{d\mathbf{v}_{n-1}}{ds} &= -\kappa_{n-2} \mathbf{v}_{n-2}.\end{aligned}$$

The curvature functions κ_j and the orthonormal vectors \mathbf{v}_j have the following coordinate expressions

$$\kappa_j = \sqrt{\frac{(\mathbf{x}_\perp^{(j+1)})^2}{\dot{\mathbf{x}}^2 (\mathbf{x}_\perp^{(j)})^2}}, \quad \mathbf{v}_j = \frac{\mathbf{x}_\perp^{(j)}}{\sqrt{(\mathbf{x}_\perp^{(j)})^2}},$$

and $\mathbf{x}_\perp^{(j)}$ is defined recursively as follows:

$$\mathbf{x}_\perp^{(j)} = \mathbf{x}^{(j)} - \sum_{i=1}^{j-1} \frac{\mathbf{x}^{(j)} \cdot \mathbf{x}_\perp^{(i)}}{(\mathbf{x}_\perp^{(i)})^2} \mathbf{x}_\perp^{(i)}.$$

The condition that $\mathbf{x}_\perp^{(j)}$ be different from zero for $j = 1, \dots, n-1$ is tantamount to saying that the curve γ is nicely curved in \mathbf{R}^{n-1} , *i.e.* that all curvatures $\kappa_1, \dots, \kappa_{n-1}$ are nowhere vanishing functions and that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ form an orthonormal basis.

Accordingly we can describe an arbitrary deformation of the curve γ as

$$\delta_\eta \mathbf{x} = \sum_{j=1}^{n-1} \eta_j \mathbf{x}_\perp^{(j)}.$$

A detailed proof that there is a particular parametrization of such transformations that reproduces the W_n -algebra is beyond the scope of this note, nevertheless we will sketch here the required steps. The main observation is that the identity

$$\mathbf{x}^{(n)} = \sum_{j=1}^{n-1} \frac{\mathbf{x}^{(n)} \cdot \mathbf{x}_\perp^{(j)}}{(\mathbf{x}_\perp^{(j)})^2} \mathbf{x}_\perp^{(j)}$$

can be rewritten in the form $\mathsf{L}\mathbf{x} = 0$, with L a differential operator of the form

$$\mathsf{L} = \frac{d^n}{dt^n} - \sum_{j=1}^{n-1} u_j \frac{d^j}{dt^j},$$

by expressing all (uncontracted) vectors $\mathbf{x}_\perp^{(i)}$ in terms of $\mathbf{x}^{(k)}$.

Just as in the two-dimensional case, the einbein $e = \sqrt{\dot{\mathbf{x}}^2}$ and the curvature functions κ_i determine a parametrized curve up to Euclidean motions. On the other hand, the coefficients u_j are scalar functions made out of the derivatives of $\mathbf{x}(t)$, so they are insensitive to Euclidean motions. It is clear then that their expression is uniquely determined in terms of e and κ_i . In particular, it can be shown that

$$u_{n-1} = \frac{\mathbf{x}^{(n)} \mathbf{x}_\perp^{(n-1)}}{(\mathbf{x}_\perp^{(n-1)})^2}$$

is a total derivative. Indeed, from the coordinate expression of Frenet equations,

$$\frac{d\mathbf{x}_\perp^{(i)}}{dt} = \mathbf{x}_\perp^{(i+1)} + \left(\frac{d}{dt} \log \sqrt{(\mathbf{x}_\perp^{(i)})^2} \right) \mathbf{x}_\perp^{(i)} - \frac{(\mathbf{x}_\perp^{(i)})^2}{(\mathbf{x}_\perp^{(i-1)})^2} \mathbf{x}_\perp^{(i-1)},$$

we can recursively express all $\mathbf{x}^{(k)}$ in terms of the $\mathbf{x}_\perp^{(j)}$ with $1 < j \leq k$. A direct computation now yields

$$u_{n-1} = \frac{d}{dt} \left(\sum_{i=1}^{n-1} \log \sqrt{(\mathbf{x}_\perp^{(i)})^2} \right).$$

Incidentally, the same procedure can be followed to get the explicit expressions for all other u_j in terms of $\sqrt{(\mathbf{x}_\perp^{(i)})^2}$ and hence in terms of the einbein and the curvatures.

Now, a rescaling of \mathbf{x}

$$\mathbf{x} \longrightarrow \left(\prod_{i=1}^{n-1} (\mathbf{x}^{(i)})^2 \right)^{\frac{1}{2n}} \mathbf{x}$$

allows us to eliminate this term and re-express L in the standard form:

$$\mathsf{L} = \frac{d^n}{dt^n} + \sum_{j=0}^{n-2} W_j \frac{d^j}{dt^j}.$$

It is now possible to use the results of [5], where it is shown that the algebra of deformations of L preserving its form is nothing but a classical version of W_n .

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REFERENCES

- [1] A. B. Zamolodchikov, *Theor. Mat. Phys.* **65** (1986) 1205.
- [2] P. Bowcock and G.M.T. Watts, *Nucl. Phys.* **B379** (1992) 63 ([hep-th/9111062](#)).
- [3] G. Sotkov and M. Stanishkov, *Nucl. Phys.* **B356** (1991) 439; G. Sotkov, M. Stanishkov and C.J. Zhu, *Nucl. Phys.* **B356** (1991) 245.
J.L. Gervais and Y. Matsuo, *Phys. Lett.* **274B** (1992) 309 ([hep-th/9110028](#)); *Comm. Math. Phys.* **152** (1993) 317 ([hep-th/9201026](#)).
J.M. Figueroa-O'Farrill, E. Ramos and S. Stanciu, *Phys. Lett.* **297B** (1992) 289, ([hep-th/9209002](#)).
W.S. l'Yi and O.A. Soloviev, *Nucl. Phys.* **B397** (1993) 417.
J. Gomis, J. Herrero, K. Kamimura and J. Roca, *Phys. Lett.* **339B** (1994) 59 ([hep-th/9409024](#)).
- [4] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, *Nucl. Phys.* **B349** (1991) 791,
C.M. Hull, *Comm. Math. Phys.* **156** (1993) 245,
J. De Boer and J. Goeree, *Nucl. Phys.* **B401** (1993) 369, ([hep-th/9206098](#))
S. Govindarajan and T. Jayaraman, “A proposal for the geometry of W_n gravity”,
([hep-th/9405146](#)).
- [5] A.O. Radul, *Sov. Phys. JETP Lett.* **50** (1989) 371; *Functional Analysis and Its Application* **25** (1991) 25.